

# A note on Gekeler's $h$ -function

Florian Breuer \*

Stellenbosch University, Stellenbosch, South Africa  
fbreuer@sun.ac.za

May 10, 2016

## Abstract

We give a brief introduction to Drinfeld modular forms, concentrating on the many equivalent constructions of the form  $h$  of weight  $q + 1$  and type 1, to which we contribute some new characterizations involving Moore determinants, and an application to the Weil pairing on Drinfeld modules. We also define Drinfeld modular functions of non-zero type and provide a moduli interpretation of these.

## 1 Drinfeld modular forms

We start with a brief introduction to Drinfeld modular forms, see e.g. [7] for more details.

Let  $\mathbb{F}_q$  denote the finite field of order  $q$ , and set  $A = \mathbb{F}_q[T]$ , the ring of polynomials over  $\mathbb{F}_q$ . We furthermore set  $K = \mathbb{F}_q(T)$ ,  $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ , the completion of  $K$  at the infinite place, and  $\mathbb{C}_\infty = \hat{K}_\infty$ , the completion of an algebraic closure of  $K_\infty$ , which is an algebraically closed complete non-Archimedean field. The rings  $A, K, K_\infty$  and  $\mathbb{C}_\infty$  are the function field analogues of the more usual  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$ .

An  $A$ -lattice  $\Lambda \subset \mathbb{C}_\infty$  of rank  $r \geq 1$  is an  $A$ -submodule of the form  $\Lambda = \omega_1 A + \omega_2 A + \cdots + \omega_r A$ , where  $\omega_1, \omega_2, \dots, \omega_r \in \mathbb{C}_\infty$  are linearly independent over  $K_\infty$ . To such a lattice we associate its *exponential function*

$$e_\Lambda(x) = x \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{x}{\lambda}\right),$$

and  $e_\Lambda : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  is holomorphic, surjective,  $\Lambda$ -periodic and  $\mathbb{F}_q$ -linear, with simple zeroes on  $\Lambda$ . It is the analogue of the usual exponential function when  $r = 1$  and of the elliptic functions when  $r = 2$ . Since  $e_\Lambda$  is  $\mathbb{F}_q$ -linear, its logarithmic derivative is

$$\frac{1}{e_\Lambda(x)} = \sum_{\lambda \in \Lambda} \frac{1}{x + \lambda}.$$

Denote by  $\mathbb{C}_\infty\{X^q\} = \{a_0 X + a_1 X^q + \cdots + a_n X^{q^n} \mid n \geq 0, a_0, \dots, a_n \in \mathbb{C}_\infty\}$  the non-commutative ring of  $\mathbb{F}_q$ -linear polynomials over  $\mathbb{C}_\infty$ , where multiplication is defined via composition of polynomials.

For each  $a \in A$  the exponential function satisfies the functional equation

$$e_\Lambda(ax) = \varphi_a^\Lambda(e_\Lambda(x)),$$

where  $\varphi_a^\Lambda(X) \in \mathbb{C}_\infty\{X^q\}$  has degree  $q^{r \deg a}$ . The map

$$A \longrightarrow \mathbb{C}_\infty\{X\}; \quad a \longmapsto \varphi_a^\Lambda(X)$$

---

\*Supported by grant no. IFRR96241 of the National Research Foundation of South Africa

is an  $\mathbb{F}_q$ -algebra monomorphism called a *Drinfeld module of rank  $r$* , and plays the role of  $\mathbb{G}_m$  in rank  $r = 1$  and of elliptic curves when  $r = 2$ . There seems to be no classical analogue for Drinfeld modules of rank  $r \geq 3$ . More information on Drinfeld modules can be found in [10, Chapter 4].

The archetypal rank 1 Drinfeld module is the *Carlitz module*, defined by

$$\rho_T(X) = TX + X^q.$$

It corresponds to the lattice  $\bar{\pi}A$ , where  $\bar{\pi} \in \mathbb{C}_\infty$  is defined up to a factor in  $\mathbb{F}_q^*$ , is transcendental over  $K$ , and plays the role of the classical  $2\pi i \in \mathbb{C}$ . Any two rank 1 lattices in  $\mathbb{C}_\infty$  are homothetic, and consequently any two rank 1 Drinfeld modules are isomorphic over  $\mathbb{C}_\infty$ .

In the case of rank  $r = 2$ , every lattice is homothetic to a lattice of the form  $zA + A$ , where  $z$  lies in the Drinfeld period domain

$$\Omega := \mathbb{C}_\infty - K_\infty,$$

which is analogous to the classical upper half-plane. The group  $\Gamma = \mathrm{GL}_2(A)$  acts on  $\Omega$  via fractional linear transformations,

$$\gamma(z) = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Analogous to the classical situation, we can define: a *Drinfeld modular form* for  $\Gamma$  of weight  $k \in \mathbb{Z}_+$  and type  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$  is a holomorphic function  $f : \Omega \rightarrow \mathbb{C}_\infty$  satisfying

- (a)  $f(\gamma z) = \det(\gamma)^{-m}(cz + d)^k f(z)$  for all  $\gamma \in \Gamma$ , and
- (b)  $f(z)$  is holomorphic at infinity.

This second condition can be understood as follows. From (a) follows that  $f(z+1) = f(z)$ , hence it admits a “Fourier series”

$$f(z) = \sum_{n \in \mathbb{Z}} a_n t(z)^n, \quad a_n \in \mathbb{C}_\infty,$$

where

$$t(z) := \frac{1}{e_{\bar{\pi}A}(\bar{\pi}z)} = \frac{1}{\bar{\pi}e_A(z)} = \bar{\pi}^{-1} \sum_{a \in A} \frac{1}{z + a}$$

is the parameter at infinity analogous to the classical  $\exp(2\pi iz)$ . Now (b) asserts that in the above expansion,  $a_n = 0$  for all  $n < 0$ . If additionally  $a_0 = 0$ , we call  $f$  a *cusp form*.

If we set  $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$  with  $\alpha \in \mathbb{F}_q^*$ , then the transformation rule (a) gives  $f(z) = \alpha^{k-2m} f(z)$ , so non-zero modular forms for  $\Gamma$  can only exist if the type and weight satisfy  $k \equiv 2m \pmod{q-1}$ .

The set  $M_{k,m}(\Gamma)$  of Drinfeld modular forms of weight  $k$  and type  $m$  forms a finite dimensional  $\mathbb{C}_\infty$ -vector space.

As a first example of such forms, for  $z \in \Omega$  we define the rank 2 lattice

$$\Lambda_z = \bar{\pi}(zA + A) \subset \mathbb{C}_\infty$$

and denote the associated rank 2 Drinfeld module by  $\varphi^z$ . It is determined by

$$\varphi_T^z(X) = TX + g(z)X^q + \Delta(z)X^{q^2},$$

whose coefficients  $g(z)$  and  $\Delta(z)$  are Drinfeld modular forms of type 0 and weights  $q-1$  and  $q^2-1$ , respectively.

The factor  $\bar{\pi}$  in the definition of  $\Lambda_z$  has been included so that the Fourier coefficients of  $g$  and  $\Delta$  turn out to be elements of  $A$ . This normalization is consistent with the second half of [5]; the reader is warned that normalization conventions vary across the literature.

It turns out that  $\Delta$  is a cusp form and David Goss [9] has shown that the graded ring of Drinfeld modular forms of type 0 is generated by  $g$  and  $\Delta$ :

$$\bigoplus_{k \geq 1} M_{k,0}(\Gamma) = \mathbb{C}_\infty[g, \Delta]$$

and  $g$  and  $\Delta$  are algebraically independent over  $\mathbb{C}_\infty$ .

## 2 The many faces of $h$

In this note, we are interested in Drinfeld modular forms of type 1 and weight  $q + 1$ , the first instance of forms with non-zero type. It is known that  $M_{q+1,1}(\Gamma)$  is one-dimensional, so there is only one such form, up to a constant multiple, and it is traditionally denoted  $h$ . It is defined as the Poincaré series

$$h(z) := \sum_{\gamma \in H \backslash \Gamma} \det(\gamma)(cz + d)^{-q-1} t(\gamma z), \quad (2.1)$$

where  $H$  is the subgroup of  $\Gamma$  of elements of the form  $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ . The earliest appearance of such series in the literature appears to be in [8, Chapter X].

In his seminal paper [5], Ernst-Ulrich Gekeler proved that  $h$  is a nowhere vanishing cusp form for  $\Gamma$  of weight  $q + 1$  and type 1, computed its first few Fourier coefficients and proved that the graded ring of modular forms of arbitrary type is generated by  $g$  and  $h$ :

$$\bigoplus_{k \geq 1, m \in \mathbb{Z}/(q-1)\mathbb{Z}} M_{k,m}(\Gamma) = \mathbb{C}_\infty[g, h], \quad (2.2)$$

where  $g$  and  $h$  are algebraically independent.

Gekeler also proved a number of other characterizations of  $h$ , such as

$$h(z) = \bar{\pi}^{-1} \left( \frac{d}{dz} - (q-1) \frac{\Delta'(z)}{\Delta(z)} \right) g(z), \quad (2.3)$$

the *Serre derivative* of  $g$ , and

$$h(z)^{q-1} = -\Delta(z). \quad (2.4)$$

Thus, from his product formula [4] for  $\Delta$ ,

$$h(z) = -t(z) \prod_{a \in A_+} f_a(t(z))^{q^2-1}, \quad (2.5)$$

where  $A_+$  denotes the monic elements of  $A$  and  $f_a(X) = X^{q^{\deg(a)}} \rho_a(X^{-1})$ . See also [3, (6.2)].

Furthermore,

$$h(z) = -\bar{\pi}^{-q} \left| \begin{matrix} z & 1 \\ \eta_1(z) & \eta_2(z) \end{matrix} \right|^{-1}, \quad (2.6)$$

where  $\eta_1$  and  $\eta_2$  are certain quasi-periodic functions [6]. This is an analogue of the Legendre period relation, and links up with the De Rham cohomology of Drinfeld modules.

Finally, another attractive characterization of  $h$  is via its *A-expansion*, due to Bartolomé López [11]:

$$h(z) = - \sum_{a \in A_+} a^q t(az). \quad (2.7)$$

## 3 $T$ -torsion and Moore determinants

Let  $V = (T^{-1}A/A)^2 \cong \mathbb{F}_q^2$ . Then for each  $v = (v_1 T^{-1}, v_2 T^{-1}) \in V' = V - \{0\}$  we obtain a weight one Eisenstein series for  $\Gamma(T) = \ker(\Gamma \rightarrow \mathrm{GL}_2(\mathbb{F}_q))$ ,

$$\begin{aligned} E_v(z) &:= \bar{\pi}^{-1} \sum_{\substack{a, b \in T^{-1}A \\ (a, b) \equiv v \pmod{A^2}}} \frac{1}{az + b} \\ &= \frac{1}{\bar{\pi} e_{zA+A}(T^{-1}(v_1 z + v_2))} = \frac{1}{e_{\Lambda_z}(\bar{\pi} T^{-1}(v_1 z + v_2))}. \end{aligned}$$

Their reciprocals are the non-zero  $T$ -torsion points of  $\varphi^z$ , i.e. we have

$$\varphi_T^z(X) = TX + g(z)X^q + \Delta(z)X^{q^2} = TX \prod_{v \in V'} (1 - E_v(z)X).$$

In particular, we obtain

$$\Delta(z) = T \prod_{v \in V'} E_v(z). \quad (3.1)$$

Gunther Cornelissen has proved [2] that in fact the ring of modular forms for  $\Gamma(T)$  is generated by the  $E_u$ ,

$$\bigoplus_{k \geq 1} M_k(\Gamma(T)) = \mathbb{C}_\infty[E_u \mid u \in V'];$$

in this case there exist algebraic relations between these generators.

In [5, (9.3)], Gekeler showed that

$$h(z) = c \sum_{u, v \in V, \langle u, v \rangle = 1} E_u^q(z) E_v(z), \quad (3.2)$$

for some non-zero constant  $c \in \mathbb{C}_\infty^*$ , where  $\langle \cdot, \cdot \rangle$  denotes a non-degenerate alternating form on  $V$ .

Our goal is to deduce some more characterizations of  $h$  along these lines.

Recall that the *Moore determinant* (see [10, Chapter 1.3]) of elements  $x_1, x_2, \dots, x_n$  of a field containing  $\mathbb{F}_q$  is defined by

$$M(x_1, x_2, \dots, x_n) = \det \left( x_i^{q^{j-1}} \right)_{1 \leq i \leq n, 1 \leq j \leq n}.$$

Now choose an ordered basis  $u, v$  for  $V$ , then

$$\begin{aligned} \varphi_T^z(X) &= T M(E_u(z)^{-1}, E_v(z)^{-1}, X) / M(E_u(z)^{-1}, E_v(z)^{-1})^q \\ &= \Delta(z) M(E_u(z)^{-1}, E_v(z)^{-1}, X) / M(E_u(z)^{-1}, E_v(z)^{-1}), \end{aligned}$$

this being the unique polynomial with roots  $\mathbb{F}_q E_u(z)^{-1} + \mathbb{F}_q E_v(z)^{-1}$ , linear term  $TX$  and leading coefficient  $\Delta(z)$ , from which we obtain

$$\Delta(z) = T M(E_u(z)^{-1}, E_v(z)^{-1})^{1-q} \quad (3.3)$$

and thus, by (2.4),

$$h(z) = {}^{q-1}\sqrt{-T} M(E_u(z)^{-1}, E_v(z)^{-1})^{-1}. \quad (3.4)$$

Here  ${}^{q-1}\sqrt{-T}$  denotes a  $(q-1)$ th root of  $-T$ , which we determine next.

Let  $u = (u_1 T^{-1}, u_2 T^{-1})$ ,  $v = (v_1 T^{-1}, v_2 T^{-1}) \in V'$ . Since the mapping  $w \mapsto E_w(z)^{-1}$  is  $\mathbb{F}_q$ -linear, we have

$$\begin{aligned} M(E_u(z)^{-1}, E_v(z)^{-1}) &= - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} M(E_{(0, T^{-1})}(z)^{-1}, E_{(T^{-1}, 0)}(z)^{-1}) \\ &= - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} E_{(0, T^{-1})}(z)^{-1} \prod_{\varepsilon \in \mathbb{F}_q} E_{(T^{-1}, \varepsilon T^{-1})}(z)^{-1}, \end{aligned}$$

by the Moore Determinant Formula [10, Cor. 1.3.7].

From [3, (2.1)] we obtain the following expansions of  $E_u(z)$  in terms of the parameter

$$t_T = t_T(z) := \frac{1}{e_{\pi A}(T^{-1} \pi z)} = \frac{1}{\pi e_A(T^{-1} z)} = t(T^{-1} z)$$

at the cusp  $\infty$  of  $\Gamma(T) \backslash \Omega$ :

$$\begin{aligned} E_{(0, T^{-1})}(z) &= \lambda_T^{-1} + o(t_T), \\ E_{(T^{-1}, \varepsilon T^{-1})}(z) &= t_T(z) + o(t_T^2), \quad (\varepsilon \in \mathbb{F}_q), \end{aligned}$$

where

$$\lambda_T = \bar{\pi} e_A(T^{-1}) \in \rho[T]$$

is a specific  $(q-1)$ th root of  $-T$ .

Comparing this with (e.g. from (2.5))

$$h(z) = -t(z) + o(t^2) = -t_T(z)^q + o(t_T^{q+1}),$$

we obtain

**Theorem 3.1.** *Let  $u = (u_1 T^{-1}, u_2 T^{-1})$ ,  $v = (v_1 T^{-1}, v_2 T^{-1}) \in V'$  and  $\lambda_T = \bar{\pi} e_A(T^{-1}) \in \rho[T]$ . The following relations hold:*

$$M(E_u(z)^{-1}, E_v(z)^{-1}) = \lambda_T \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} h(z)^{-1}, \quad \text{and} \quad (3.5)$$

$$h(z) = \lambda_T E_{(0, T^{-1})}(z) \prod_{\varepsilon \in \mathbb{F}_q} E_{(T^{-1}, \varepsilon T^{-1})}(z) \quad (3.6)$$

$$= c \lambda_T \prod_{w \in \mathbb{P}(V)} E_w(z), \quad (3.7)$$

where the last product runs over a set of representatives in  $V$  of the projective line  $\mathbb{P}(V) \cong \mathbb{P}^1(\mathbb{F}_q)$ , and  $c \in \mathbb{F}_q^*$  depends on the choice of these representatives.  $\square$

Note that we can also obtain (3.6) and (3.7), up to a multiplicative constant, from (2.4) and (3.1).

## 4 The Weil pairing

The *determinant* of a rank 2 Drinfeld module  $\varphi_T(X) = TX + gX^q + \Delta X^{q^2}$  is the rank 1 Drinfeld module defined by

$$\psi_T(X) = TX - \Delta X^q,$$

and for each  $a = a_0 + a_1 T + \cdots + a_n T^n \in A$  the *Weil pairing* [13, Prop. 7.4] is given by

$$\begin{aligned} w_a : \varphi[a] \times \varphi[a] &\longrightarrow \psi[a], \\ (x, y) &\longmapsto \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} a_{i+j+1} M(\varphi_{T^j}(x), \varphi_{T^i}(y)). \end{aligned} \quad (4.1)$$

In the case  $a = T$ , the Weil pairing is particularly simple – it is the Moore determinant:  $w_T(x, y) = M(x, y) = xy^q - x^q y$ . In this light, we see that (2.4) and (3.5) each imply

$$\psi_T^z(\lambda_T h(z)^{-1}) = 0. \quad (4.2)$$

The determinant of  $\varphi^z$  is isomorphic to the Carlitz module via

$$\psi^z = h(z)^{-1} \cdot \rho \cdot h(z), \quad (4.3)$$

so its associated lattice is  $L_z = \bar{\pi} h(z)^{-1} A$ , and its  $a$ -torsion module is generated by  $\bar{\pi} e_A(\frac{1}{a}) h(z)^{-1}$ .

When  $a = T$ , (3.5) gives an explicit expression for the Weil pairing on  $\varphi^z$ . In general, we have

**Theorem 4.1.** *Let  $a \in A$ . The Weil pairing on  $\varphi^z[a]$  is given by*

$$w_a\left(e_{\Lambda_z}(\bar{\pi}a^{-1}(u_1z + u_2)), e_{\Lambda_z}(\bar{\pi}a^{-1}(v_1z + v_2))\right) = \bar{\pi}e_A(a^{-1}(u_1v_2 - u_2v_1))h(z)^{-1}. \quad (4.4)$$

*Proof.* It suffices to show this for  $(u_1, u_2) = (1, 0)$  and  $(v_1, v_2) = (0, 1)$ , since the general case follows from the alternating property of the Weil pairing.

Both sides lie in  $\psi^z[a]$ , hence are equal up to a multiplicative constant. This constant can be computed by comparing the first coefficients of their  $t_a(z) := \bar{\pi}^{-1}e_A(a^{-1}z)$ -expansions, as in the previous section.  $\square$

It is instructive to compare this with the analogous expression for the Weil pairing on an elliptic curve  $E = \mathbb{C}/(z\mathbb{Z} + \mathbb{Z})$ , see [12, Ex. 1.15, p.89],

$$w_N(N^{-1}(u_1z + u_2), N^{-1}(v_1z + v_2)) = e^{2\pi i N^{-1}(u_1v_2 - u_2v_1)},$$

where the right hand side does not depend on  $z$ . The reason for this is that there is only one  $\mathbb{G}_m$ , whereas there are many rank 1 Drinfeld modules; the factor  $h(z)^{-1}$  in (4.4) serves to pick out the correct one, i.e.  $\psi^z$ .

## 5 Modular functions of non-zero type

A *modular function* of type  $m$  is a meromorphic function  $f : \Omega \rightarrow \mathbb{C}_\infty$  satisfying

- (a)  $f(\gamma z) = \det(\gamma)^{-m} f(z)$  for all  $\gamma \in \Gamma$ , and
- (b)  $f(z)$  is meromorphic at infinity, i.e.  $a_n = 0$  for  $n \ll 0$  in the Fourier expansion of  $f(z)$ .

As before, non-zero modular functions must satisfy  $2m \equiv 0 \pmod{q-1}$ .

It is well-known that the field of modular functions of type 0 is the rational function field over  $\mathbb{C}_\infty$  generated by

$$j(z) = \frac{g(z)^{q+1}}{\Delta(z)}.$$

**Theorem 5.1.** *The field of modular functions of arbitrary type is the rational function field  $\mathbb{C}_\infty(\tilde{j})$  over  $\mathbb{C}_\infty$  generated by the modular function*

$$\tilde{j}(z) := \frac{g(z)^{m(q+1)/(q-1)}}{h(z)^m}, \quad (5.1)$$

of type  $-m$ , where  $m$  is the least positive integer for which  $q-1 \mid m(q+1)$ .

If  $q$  is even, then  $m = q-1$  and  $\tilde{j}(z) = j(z)$ , whereas if  $q$  is odd, we have  $m = (q-1)/2$  and  $\tilde{j}(z)^2 = j(z)$ . The function  $\tilde{j}$  is holomorphic on  $\Omega$  with a pole of order  $m$  at  $\infty$ .

*Proof.* Let  $f$  be a modular function, with poles  $z_1, \dots, z_n \in \Omega$ . Then for  $l$  sufficiently large,  $f(z)h(z)^l \prod_{i=1}^n (j(z) - j(z_i))^l$  is a modular form of weight  $l(q+1)$ , hence by (2.2) equals  $F[g, h]$ , where  $F[X, Y] \in \mathbb{C}_\infty[X, Y]$  is a homogeneous polynomial of weighted degree  $l(q+1)$ , where  $X$  and  $Y$  are assigned weights of  $q-1$  and  $q+1$ , respectively. The result now follows easily.  $\square$

Suppose from now on that  $q$  is odd, and consider the map

$$\pi : \Omega \longrightarrow \mathbb{P}_{\mathbb{C}_\infty}(q-1, q+1); \quad z \longmapsto [g(z) : h(z)],$$

where  $\mathbb{P}_{\mathbb{C}_\infty}(q-1, q+1)$  is the weighted projective space [1] over  $\mathbb{C}_\infty$  with weights  $(q-1, q+1)$ ; this is the set of  $\mathbb{C}_\infty$ -valued points on  $\text{Proj } \mathbb{C}_\infty[g, h]$ .

Denote by  $\Gamma_2$  the subgroup of matrices in  $\Gamma$  with square determinant,

$$\Gamma_2 = \{\gamma \in \Gamma \mid \det(\gamma) \in \mathbb{F}_q^{*2}\}.$$

We easily compute that, for  $z_1, z_2 \in \Omega$ , we have  $\pi(z_1) = \pi(z_2)$  if and only if  $z_1 = \gamma(z_2)$  with  $\gamma \in \Gamma_2$  if  $g(z_1) \neq 0$  and  $\gamma \in \Gamma$  if  $g(z_1) = 0$ .

The image of  $\pi$  corresponds to the open affine where  $h \neq 0$ , which is

$$\text{Spec}(\mathbb{C}_\infty[g, h, h^{-1}])_{\deg=0} = \text{Spec} \mathbb{C}_\infty[\tilde{j}].$$

Thus we see that  $\tilde{j}$  induces a map

$$\tilde{j} : \Gamma_2 \backslash \Omega \longrightarrow \mathbb{C}_\infty$$

which is bijective except above 0, which has two pre-images.

Lastly, we describe a moduli interpretation of  $\tilde{j}$ : when  $\tilde{j} \neq 0$  it parametrizes isomorphism classes of Drinfeld modules decorated with  $\mathbb{F}_q^{*2}$ -classes of  $T$ -torsion points on their determinant modules.

More precisely, let  $F$  be an algebraically closed extension of  $K$  and fix  $0 \neq \lambda_T \in \rho[T] \subset F$ , a non-zero  $T$ -torsion point of the Carlitz module (equivalently, a  $(q-1)$ th root of  $-T$ ). We consider pairs  $(\varphi, \lambda)$ , where  $\varphi$  is a rank 2 Drinfeld module defined by  $\varphi_T(X) = TX + gX^q + \Delta X^{q^2}$  over  $F$  and  $0 \neq \lambda \in \psi[T]$  is a non-zero  $T$ -torsion point of its determinant module  $\psi$ . To such a pair we associate its  $\tilde{j}$ -invariant

$$\tilde{j} = (\lambda \lambda_T^{-1})^{(q-1)/2} g^{(q+1)/2} \in F. \quad (5.2)$$

We call two such pairs,  $(\varphi, \lambda)$  and  $(\varphi', \lambda')$ , isomorphic if either  $g = g' = 0$  or if  $g \neq 0$  and there exist  $c \in \mathbb{C}_\infty^*$  and  $\varepsilon \in \mathbb{F}_q^{*2}$  such that  $\varphi' = c \cdot \varphi \cdot c^{-1}$  and  $\lambda' = \varepsilon c \lambda$ . As simple computation shows that two such pairs are isomorphic over  $F$  if and only if their  $\tilde{j}$ -invariants are equal. We have shown

**Theorem 5.2.** *The  $F$ -valued points of  $\text{Spec } F[\tilde{j}]$  parametrize isomorphism classes of pairs  $(\varphi, \lambda)$  described above.  $\square$*

The  $j$ -line  $\text{Spec } F[j]$  parametrizes isomorphism classes of Drinfeld modules, and the forgetful functor gives the double cover  $\text{Spec } F[\tilde{j}] \rightarrow \text{Spec } F[j]$ , ramified above 0, as  $\tilde{j}^2 = j$ .

Lastly, a point  $[g : h] \in \mathbb{P}_F(q-1, q+1)$  defines (up to isomorphism) a pair  $(\varphi, \lambda)$  with

$$\begin{aligned} \varphi_T(X) &= TX + gX^q - h^{q-1}X^{q^2}, \\ \psi_T(X) &= TX + h^{q-1}X^{q-1}, \\ \lambda &= \lambda_T h^{-1} \in \psi[T], \\ \tilde{j}(\varphi, \lambda) &= g^{(q+1)/2} / h^{(q-1)/2} \end{aligned}$$

and moreover, every pair  $(\varphi, \lambda)$  arises in this way.

## References

- [1] Mauro Beltrametti and Lorenzo Robbiano. *Introduction to the theory of weighted projective spaces*. Expos. Math. **4**(2) (1986), 111–162.
- [2] Gunther Cornelissen. *Drinfeld modular forms of level  $T$* . In *Drinfeld modules, modular schemes and applications* (Alden-Biesen, 1996), pages 272–281. World Sci. Publ., River Edge, NJ, 1997.
- [3] Ernst-Ulrich Gekeler. *Modulare Einheiten für Funktionenkörper*. J. Reine Angew. Math. **348** (1984), 94–115.
- [4] Ernst-Ulrich Gekeler. *A product expansion for the discriminant function of Drinfel'd modules of rank two*. J. Number Theory **21**(2) (1985), 135–140.
- [5] Ernst-Ulrich Gekeler. *On the coefficients of Drinfel'd modular forms*. Invent. Math. **93**(3) (1988), 667–700.

- [6] Ernst-Ulrich Gekeler. *Quasi-periodic functions and Drinfel'd modular forms*. Compos. Math. **69(3)** (1989), 277–293.
- [7] Ernst-Ulrich Gekeler. *A survey on Drinfeld modular forms*. Turkish J. Math. **23(4)** (1999), 485–518.
- [8] Lothar Gerritzen and Marius van der Put. *Schottky groups and Mumford curves*, volume 817 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [9] David Goss. *Modular forms for  $\mathbb{F}_r[T]$* . J. Reine Angew. Math. **317** (1980), 16–39.
- [10] David Goss. *Basic structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1996.
- [11] Bartolomé López. *A non-standard Fourier expansion for the Drinfeld discriminant function*. Arch. Math. **95** (2010), 143–150.
- [12] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate texts in Mathematics*. Springer-Verlag, Berlin, 1994.
- [13] Gert-Jan van der Heiden. *Weil pairing for Drinfeld modules*. Monatsh. Math. **143(2)** (2004), 115–143.